数列 $\left(1+\frac{x}{n}\right)^n$ の e^x に対する近似度
A Degree of the Approximation for e^x by the Sequence $\left(1+\frac{x}{n}\right)^n$

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概 要

Abstract

We give here a degree of the approximation for e^x by the sequence $(1+\frac{x}{n})^n$ $(n \in \mathbf{N})$.

Key words ; Napier's constant, Euler's number, e.
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次の定理を証明する。

定理 1　$0 < 2x < n \in \mathbf{N}$ のとき，

$$1 > \frac{\left(1+\frac{x}{n}\right)^n}{e^x} > e^{-\frac{x^2}{2n}}.$$

系 1　$0 < 2x < n \in \mathbf{N}$ のとき，

$$0 < e^x - \left(1+\frac{x}{n}\right)^n < e^x\left(1-e^{-\frac{x^2}{2n}}\right) \text{ or}$$

$0 < x < \sqrt{n}$のとき，

$$0 < e^x - \left(1 + \frac{x}{n}\right)^n < \frac{x^2}{2n} e^x \ .$$

定理１の証明

e^x, $\left(1+\frac{x}{n}\right)^n$ の log を取った差：

$$x - n\log\left(1 + \frac{x}{n}\right)$$

の Maclaurin-Taylor 展開を考えると

$$\begin{aligned}
& x - n\log\left(1 + \frac{x}{n}\right) = \\
&= x - n\left\{\frac{x}{n} - \frac{1}{2}\left(\frac{x}{n}\right)^2 + \frac{1}{3}\left(\frac{x}{n}\right)^3 - \frac{1}{4}\left(\frac{x}{n}\right)^4 + \frac{1}{5}\left(\frac{x}{n}\right)^5 - \cdots\right\} = \\
&= \frac{1}{2}\frac{x^2}{n} - \frac{1}{3}\frac{x^3}{n^2} + \frac{1}{4}\frac{x^4}{n^3} - \frac{1}{5}\frac{x^5}{n^4} + \cdots = \\
&= \frac{x^2}{2n}\left\{1 - \frac{2}{3}\frac{x}{n} + \frac{2}{4}\left(\frac{x}{n}\right)^2 - \frac{2}{5}\left(\frac{x}{n}\right)^3 + \frac{2}{6}\left(\frac{x}{n}\right)^4 - \cdots\right\} = \\
&= \frac{x^2}{2n}\left[1 - \frac{2}{3}\frac{x}{n}\left\{1 - \frac{3}{4}\left(\frac{x}{n}\right) + \frac{3}{5}\left(\frac{x}{n}\right)^2 - \frac{3}{6}\left(\frac{x}{n}\right)^3 + \frac{3}{7}\left(\frac{x}{n}\right)^4 - \cdots\right\}\right] \\
&= \frac{x^2}{2n}\left[1 - \frac{2}{3}\frac{x}{n}\left\{1 - \frac{3}{4}\frac{x}{n}\left(1 - \frac{4}{5}\left(\frac{x}{n}\right) + \frac{4}{6}\left(\frac{x}{n}\right)^2 - \frac{4}{7}\left(\frac{x}{n}\right)^3 + \cdots\right)\right\}\right] \\
&< \frac{x^2}{2n}\left[1 - \frac{2}{3}\frac{x}{n}\left\{1 - \frac{3}{4}\frac{x}{n}\left(1 + \left(\frac{x}{n}\right) + \left(\frac{x}{n}\right)^2 + \left(\frac{x}{n}\right)^3 + \cdots\right)\right\}\right] \\
&= \frac{x^2}{2n}\left[1 - \frac{2}{3}\frac{x}{n}\left\{1 - \frac{3}{4}\left(\frac{x}{n}\right)\frac{1}{1 - \frac{x}{n}}\right\}\right] \\
&= \frac{x^2}{2n}\left[1 - \frac{2}{3}\frac{x}{n} f\left(\frac{x}{n}\right)\right] \ \cdots(*) \\
&, \ \mathit{where}\ f(y) := 1 - \frac{3y}{4(1-y)}, \ (0 < y \leq \frac{1}{2})
\end{aligned}$$

ここで

$$f'(y) = -\frac{3}{4(1-y)^2} < 0$$

であるので，上記 (*) は

$$\begin{aligned}
& (*) < \frac{x^2}{2n}\left[1 - \frac{2}{3}\frac{x}{n}\min_{0<y\leq\frac{1}{2}} f(y)\right] = \frac{x^2}{2n}\left[1 - \frac{2}{3}\frac{x}{n} f\left(\frac{1}{2}\right)\right] = \frac{x^2}{2n}\left[1 - \frac{2}{3}\frac{x}{n}\frac{1}{4}\right] = \\
&= \frac{x^2}{2n}\left[1 - \frac{1}{6}\frac{x}{n}\right]
\end{aligned}$$

従って

$$x - n\log\left(1+\frac{x}{n}\right) < \frac{x^2}{2n}\left[1-\frac{1}{6}\frac{x}{n}\right] < \frac{x^2}{2n}.$$

これを e の肩に乗せると

$$\exp\left(x - n\log\left(1+\frac{x}{n}\right)\right) < e^{\frac{x^2}{2n}}$$

$$\Longleftrightarrow$$

$$e^x\left(1+\frac{x}{n}\right)^{-n} < e^{\frac{x^2}{2n}}$$

$$\Longleftrightarrow$$

$$\frac{\left(1+\frac{x}{n}\right)^n}{e^x} > e^{-\frac{x^2}{2n}} \quad \cdots (1)$$

数列 $\left\{\left(1+\frac{x}{n}\right)^n\right\}$ は単調増加で e^x に近づく monotonically increasing $\left(1+\frac{x}{n}\right)^n \uparrow e^x$ ので

$$1 > \frac{\left(1+\frac{x}{n}\right)^n}{e^x} \quad \cdots (2).$$

(1),(2) より定理の証明は完了した。□

系 1 の証明

前半については，定理 1 より

$$e^x > \left(1+\frac{x}{n}\right)^n > e^x e^{-\frac{x^2}{2n}}$$

$$\Longleftrightarrow$$

$$0 > \left(1+\frac{x}{n}\right)^n - e^x > \left(e^{-\frac{x^2}{2n}} - 1\right)e^x$$

$$\Longleftrightarrow$$

$$0 < e^x - \left(1+\frac{x}{n}\right)^n < \left(1 - e^{-\frac{x^2}{2n}}\right)e^x.$$

となり系 1 の前半は証明された。□

後半については，前半の最右辺の $(\cdots)$ を Maclaurin-Taylor 展開して

$$\left(1-e^{-\frac{x^2}{2n}}\right) = \left(\frac{x^2}{2n}\right) - \frac{1}{2!}\left(\frac{x^2}{2n}\right)^2 + \frac{1}{3!}\left(\frac{x^2}{2n}\right)^3 - \frac{1}{4!}\left(\frac{x^2}{2n}\right)^4 + \cdots$$

$$= \left(\frac{x^2}{2n}\right)\left[1 - \frac{1}{2!}\left(\frac{x^2}{2n}\right) + \frac{1}{3!}\left(\frac{x^2}{2n}\right)^2 - \frac{1}{4!}\left(\frac{x^2}{2n}\right)^3 + \cdots\right]$$

$$= \left(\frac{x^2}{2n}\right)\left[1-\frac{1}{2!}\left(\frac{x^2}{2n}\right)\left\{1-\frac{2!}{3!}\left(\frac{x^2}{2n}\right)+\frac{2!}{4!}\left(\frac{x^2}{2n}\right)^2-\cdots\right\}\right]$$

$$= \left(\frac{x^2}{2n}\right)\times$$

$$\times\left[1-\frac{1}{2!}\left(\frac{x^2}{2n}\right)\left\{1-\frac{2!}{3!}\left(\frac{x^2}{2n}\right)\left\{1-\frac{3!}{4!}\left(\frac{x^2}{2n}\right)+\frac{3!}{5!}\left(\frac{x^2}{2n}\right)^2-\cdots\right\}\right\}\right]$$

$$< \left(\frac{x^2}{2n}\right)\times$$

$$\times\left[1-\frac{1}{2!}\left(\frac{x^2}{2n}\right)\left\{1-\frac{2!}{3!}\left(\frac{x^2}{2n}\right)\left\{1+\left(\frac{x^2}{2n}\right)+\left(\frac{x^2}{2n}\right)^2+\cdots\right\}\right\}\right]$$

$$= \left(\frac{x^2}{2n}\right)\left[1-\frac{1}{2!}\left(\frac{x^2}{2n}\right)\left\{1-\frac{1}{3}\left(\frac{x^2}{2n}\right)\frac{1}{1-\frac{x^2}{2n}}\right\}\right]\ (0<x<\sqrt{n})$$

$$= \left(\frac{x^2}{2n}\right)\left[1-\frac{1}{2!}\left(\frac{x^2}{2n}\right)f(y)\right]$$

$f(y) := 1-\frac{1}{3}\frac{y}{1-y}\ (0<y:=\frac{x^2}{2n}<\frac{1}{2})$ とする。

$f'(y) = -\frac{1}{3}\frac{1}{(1-y)^2}<0$ であるので

$$< \left(\frac{x^2}{2n}\right)\left[1-\frac{1}{2!}\left(\frac{x^2}{2n}\right)\min_{0<y\le\frac{1}{2}} f(y)\right] = \left(\frac{x^2}{2n}\right)\left[1-\frac{1}{2!}\left(\frac{x^2}{2n}\right)f(\frac{1}{2})\right]$$

$$= \left(\frac{x^2}{2n}\right)\left[1-\frac{1}{2!}\left(\frac{x^2}{2n}\right)\frac{2}{3}\right] < \left(\frac{x^2}{2n}\right)$$

これで系１の後半も証明された。□

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