トゥラン Turán の素数に関する明示公式の別証明

Another Proof of Turán's Explicit Formula for Primes

中嶋眞澄
Masumi NAKAJIMA
Department of Economics
International University of Kagoshima
Kagoshima 891-0197, JAPAN
e-mail: nakajima@eco.iuk.ac.jp

概 要

Abstract

We give here another proof of Turán's explicit formula for primes [14] and also give completion with correction for the proof of [14].
At last we give a new trucated explicit formula for primes(Main theorem) and Lemma 6 is new.
Key words ; the zeros of the Riemann Zeta-function, Turán's explicit formula for primes.
Mathematics Subject Classification 2010; 11M06, 11M26.

$\zeta(s), (s=\sigma+it \in \mathbf{C})$ を Riemann の zeta 関数とする。又, $\rho=\beta+i\gamma$ で $\zeta(s)$ の非自明な零点 non-trivial zero を表わすことにする。
$\Lambda(n)$ は von Mangoldt 関数，即ち，

$$\Lambda(n)=\begin{cases} \log p, & (n=p^k, \text{ with some prime } p \text{ and some } k \in \mathbf{N} \text{ のとき}) \\ 0, & (\text{その他のとき}) \end{cases}$$

を表わす。又，$\Gamma(w)$ は Euler の gamma 関数である。

[14] で Turán は次の素数に関する明示公式を証明しているが，それは僅か 16 行のアウトライン only 16line outline with mistakes で些細ではあるが誤りがあり訂正を要する。この公式の初出は [15] と思われるが，そこでは $\nu = 1$ のときの証明 proof only in case of $\nu = 1$ のみで, $\omega \in \mathbf{R}$ の定義が明確でなく誤りもある。他にも [16] [17] [18] [19] [20] [21] にも，この公式はあるが，いずれも証明はない。この公式は Turán が展開した素数論 Prime Nimber Theory の出発点の１つとなっている。
トゥラン Turán の素数に関する明示公式とは次のものである [14]。

トゥラン Turán の素数に関する明示公式
Turán's explicit formula for primes with error term

$s = \sigma + it,\ \sigma > 1,\ t > 0,\ X > 1,\ \nu \in \mathbf{N}$ に対して，

$$\frac{1}{\nu!}\sum_{n \geq X} \frac{\Lambda(n)}{n^s} \log^{\nu}\left(\frac{X}{n}\right) = \sum_{\rho} \frac{X^{\rho - s}}{(\rho - s)^{\nu+1}} - \frac{X^{1-s}}{(1-s)^{\nu+1}} + \mathrm{O}\left(\frac{\log(|t|+2)}{X^2}\right) \qquad \cdots (0)$$

($\sum_\rho$ は Riemann の zeta 関数 $\zeta(s)$ の非自明な零点 non-trivial zeros:ρ に亘る和) □.

[14] の証明 (outline of the proof) を完全にする complete。次の幾つかの補題が必要となる。

補題 1 ([8] 邦訳下巻 p.143, § 6.2, footnote25(訳註) の改変)
$T > c > 0,\ X > 0,\ ,\ T > 1,\ \nu \in \mathbf{N} \cup \{0\}$ に対して

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{X^w}{w^{\nu+1}} dw =$$

$$
=\begin{cases}
\frac{(\log X)^\nu}{\nu!}\delta(X)+\mathrm{O}\left(\frac{X^c}{T^\nu}\min\left\{1,\frac{1+(c-1)\delta_{\nu 0}\delta_{X1}}{\{1+(\nu-1)\delta_{X1}(1-\delta_{\nu 0})\}T^{1-\delta_{X1}(1-\delta_{\nu 0})}(|\log X|+\delta_{X1})}\right\}\right) \\
=\frac{(\log X)^\nu}{\nu!}\delta(X)+\mathrm{O}\left(\frac{X^c}{T^\nu}\min\left\{1,\frac{1}{T|\log X|}\right\}\right) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (0<X\neq 1), \\
\frac{1}{2}\delta_{\nu 0}+\mathrm{O}\left(\frac{X^c}{T^\nu}\min\left\{1,\frac{1+(c-1)\delta_{\nu 0}\delta_{X1}}{\{1+(\nu-1)\delta_{X1}(1-\delta_{\nu 0})\}T^{1-\delta_{X1}(1-\delta_{\nu 0})}(|\log X|+\delta_{X1})}\right\}\right) \\
=\frac{1}{2}\delta_{\nu 0}+\mathrm{O}\left(\frac{1}{T^\nu}\min\left\{1,\frac{1+(c-1)\delta_{\nu 0}}{\{1+(\nu-1)(1-\delta_{\nu 0})\}T^{1-(1-\delta_{\nu 0})}}\right\}\right) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (X=1), \\
=\frac{1}{2}+\mathrm{O}\left(\min\left\{1,\frac{c}{T}\right\}\right)=\frac{1}{2}+\mathrm{O}\left(\frac{c}{T}\right) \qquad (X=1,\nu=0), \\
=0+\mathrm{O}\left(\frac{1}{\nu T^\nu}\right) \qquad (X=1,\nu\in\mathbf{N})
\end{cases}
$$

$$
\delta_{xy}:=\begin{cases}1\,, & (x=y)\\ 0\,, & (x\neq y),\end{cases}\qquad \delta(X):=\begin{cases}1\,, & (1<X)\\ 0\,, & (0<X<1),\end{cases}
$$

が成立する。

注意

$0<X\neq 1,\ \nu\in\mathbf{N}$ **の場合のこの論文の補題 1** は他の文献よりも精密で良い結果である。

補題 2　[2]p.71,Theorem26

$$m<\ {}^{\exists}|T_m|<m+1\ (m=2,3,\cdots)$$

$$\text{with}\ \ \frac{\zeta'}{\zeta}(\sigma+iT_m)\ll\log^2|T_m|\ \ for\ \ -1\leq\ {}^{\forall}\sigma.$$ が成立する。

補題 3　[2]p.73,Theorem27

${}^{\forall}\sigma\leq -1,\ |{}^{\forall}(\sigma+it)-(-2q)|\geq\frac{1}{2}\ (q\in\mathbf{N})$ に対して

$$\frac{\zeta'}{\zeta}(\sigma+it)\ll\log(|\sigma+it|+1).$$

補題 4　[8] 邦訳下巻

$s\neq 1,\rho,-2q\ (q\in\mathbf{N})$ に対して

$$-\frac{\zeta'}{\zeta}(s)=$$

$$=-\log 2\pi+\left[\frac{1}{(s-1)}+\frac{1}{1}\right]-$$

$$-\sum_{q=1}^{\infty}\left[\frac{1}{s-(-2q)}+\frac{1}{(-2q)}\right]-\sum_{\rho}\left[\frac{1}{s-\rho}+\frac{1}{\rho}\right].$$

補題 5　[9] Lemma 1

$$m < {}^{\exists}|T_m| < m+1\ (m=2,3,\cdots)$$
$$\text{with}\ \frac{\zeta'}{\zeta}(\sigma+it) \ll \log^2 m$$
$$for\ -2m-1 \leq^{\forall} \sigma,\ t=T_m\ \text{ or }\ \sigma=-2m-1,\ |t|<T_m.$$

Turán's explicit formula for primes with error term の証明

$$I_\nu := \frac{1}{2\pi i}\int_{-\frac{\sigma-1}{2}-i\infty}^{-\frac{\sigma-1}{2}+i\infty} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw \cdots (1)$$

を考える。(1) に変数変換 $w=-z$ を操し $\sigma-\frac{\sigma-1}{2}=\frac{\sigma+1}{2}>1$ であることから補題 1 で $T\to\infty$ の場合を適用して項別積分をすると

$$\begin{aligned}
I_\nu &= \frac{(-1)^{\nu+1}}{2\pi i}\int_{\frac{\sigma-1}{2}-i\infty}^{\frac{\sigma-1}{2}+i\infty} -\frac{\zeta'}{\zeta}(s-z)\frac{X^{-z}}{z^{\nu+1}}dz \\
&= (-1)^{\nu+1}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^s}\frac{1}{2\pi i}\int_{\frac{\sigma-1}{2}-i\infty}^{\frac{\sigma-1}{2}+i\infty}\frac{\left(\frac{n}{X}\right)^z}{z^{\nu+1}}dz \\
&= (-1)^{\nu+1}\sum_{n\geq X}\frac{\Lambda(n)}{n^s}\frac{1}{\nu!}\log^\nu\left(\frac{n}{X}\right) \\
&= -\frac{1}{\nu!}\sum_{n\geq X}\frac{\Lambda(n)}{n^s}\log^\nu\left(\frac{X}{n}\right)\cdots(2)
\end{aligned}$$

次の 4 頂点を持つ矩形 Fig.1：

$$-\frac{\sigma-1}{2}+i(\pm T_m - t),\ (-1-\sigma)+i(\pm T_m - t)$$

([14] では $-\frac{\sigma-1}{2}\pm iT_m$, $(-1-\sigma)\pm iT_m$ となっているが，これは誤りである。)
上で正の方向の積分：

$$\begin{aligned}
&\frac{1}{2\pi i}\int_C -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw := \\
&:= \frac{1}{2\pi i}\left\{\int_{C_1}+\int_{C_2}+\int_{C_3}+\int_{C_4}\right\} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw := \\
&:= \frac{1}{2\pi i}\left\{\int_{-\frac{\sigma-1}{2}-iT_m-it}^{-\frac{\sigma-1}{2}+iT_m-it} + \int_{-\frac{\sigma-1}{2}+iT_m-it}^{(-1-\sigma)+iT_m-it} +\right.
\end{aligned}$$

$$+\int_{(-1-\sigma)+iT_m-it}^{(-1-\sigma)-iT_m-it}+\int_{(-1-\sigma)-iT_m-it}^{-\frac{\sigma-1}{2}-iT_m-it}\Bigg\}-\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw$$

$\cdots(3)$

を考える。この矩形の中にある零点，極：$\rho, -2q(q=1,2,\cdots), 1$ は

$$\begin{aligned}
-1-\sigma<\Re(\rho-s)=\beta-\sigma<-\frac{\sigma-1}{2},\\
-T_m-t<\Im(\rho-s)=\gamma-t<T_m-t,\\
-1-\sigma<\Re(-2q-s)=-2q-\sigma<-\frac{\sigma-1}{2},\\
-T_m-t<\Im(-2q-s)=-t<T_m-t,\\
-1-\sigma<\Re(1-s)=1-\sigma<-\frac{\sigma-1}{2}\\
-T_m-t<\Im(1-s)=-t<T_m-t\\
\cdots(4)
\end{aligned}$$

$$\Longleftrightarrow$$

$$\begin{aligned}
-1<\beta<\frac{\sigma+1}{2},\\
-T_m<\gamma<T_m,\\
-1<-2q<\frac{\sigma+1}{2},\\
-T_m<0<T_m,\\
-1<1<\frac{\sigma+1}{2}\\
-T_m<0<T_m\\
\cdots(4')
\end{aligned}$$

を満たさねばならない。$\sigma>1,\ 0<\beta<1$ であるから，$-1<-2q<\frac{\sigma+1}{2}$，$-T_m<\gamma<T_m$ 以外は自動的に満たされる。$q\in\mathbf{N}$ であるから，この二つのうち $-1<-2q<\frac{\sigma+1}{2}$ は如何なる q も満たさない。従って自明な零点 trivial zeros $-2q$ $(q\in\mathbf{N})$ は，この矩形の内部には存在しない。この事実から (3) に留数定理を適用すると

$$\frac{1}{2\pi i}\int_C-\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw=\frac{X^{1-s}}{(1-s)^{\nu+1}}-\sum_{|\gamma|<T_m}\frac{X^{\rho-s}}{(\rho-s)^{\nu+1}}\cdots(5)$$

次に (3) の 4 つの積分を評価する。

$$\frac{1}{2\pi i}\int_{-\frac{\sigma-1}{2}+iT_m-it}^{(-1-\sigma)+iT_m-it}-\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw=$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\int_{-\frac{\sigma-1}{2}}^{-1-\sigma} -\frac{\zeta'}{\zeta}((\sigma+u)+iT_m)\frac{X^{u+i(T_m-t)}}{\{u+i(T_m-t)\}^{\nu+1}}du \\
&\ll \int_{-\frac{\sigma-1}{2}}^{-1-\sigma}\left|\frac{\zeta'}{\zeta}(\sigma+u+iT_m)\right|\frac{X^u}{T_m^{\nu+1}}du \\
&\ll X^{-\frac{\sigma-1}{2}}\frac{\log^2 T_m}{T_m^{\nu+1}} \ll \frac{\log^2 T_m}{T_m^{\nu+1}}\cdots(6)
\end{aligned}
$$

(補題２を使った。)

同様にして

$$
\frac{1}{2\pi i}\int_{-1-\sigma+i(-T_m-t)}^{-\frac{\sigma-1}{2}+i(-T_m-t)} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw \ll \frac{\log^2 T_m}{T_m^{\nu+1}}\cdots(6')
$$

積分

$$
\frac{1}{2\pi i}\int_{(-1-\sigma)+iT_m-it}^{(-1-\sigma)-iT_m-it} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw
$$

については長くなるが初等的変形である。

$$
\begin{aligned}
&\left|\frac{1}{2\pi i}\int_{(-1-\sigma)+iT_m-it}^{(-1-\sigma)-iT_m-it} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw\right| \\
&\ll \int_{-T_m-t}^{T_m-t}\left|-\frac{\zeta'}{\zeta}(\sigma+(-1-\sigma)+i(t+v))\frac{X^{-1-\sigma}}{\{(-1-\sigma)+iv\}^{\nu+1}}\right|dv \\
&\ll X^{-1-\sigma}\int_{-\infty}^{\infty}\frac{\log(|-1+i(t+v)|+1)}{|(-1-\sigma)+iv|^{\nu+1}}dv\ \text{(補題３を使った。)} \\
&\ll X^{-1-\sigma}\int_{-\infty}^{\infty}\frac{\log(|1+ix)|+1)}{\{(1+\sigma)^2+(x-t)^2\}^{\frac{\nu+1}{2}}}dx,\ t>0,\ \sigma>1 \\
&\ll X^{-2}\int_{-\infty}^{\infty}\frac{\log(|x|+2)}{\{1+(x-t)^2\}^{\frac{\nu+1}{2}}}dx \\
&\ll X^{-2}\left\{\int_{-\infty}^{0}+\int_{0}^{t}+\int_{t}^{\infty}\right\}\frac{\log(|x|+2)}{\{1+(x-t)^2\}^{\frac{\nu+1}{2}}}dx \\
&\ll X^{-2}\left\{\int_{0}^{\infty}\frac{\log(x+2)}{\{1+(x+t)^2\}^{\frac{\nu+1}{2}}}dx+\int_{0}^{t}\frac{\log(x+2)}{\{1+(x-t)^2\}^{\frac{\nu+1}{2}}}dx+\right. \\
&\left.+\int_{t}^{\infty}\frac{\log(x+2)}{\{1+(x-t)^2\}^{\frac{\nu+1}{2}}}dx\right\} \\
&\ll X^{-2}\left\{\int_{0}^{\infty}\frac{\log(x+2)}{1+(x+t)^2}dx+\log(t+2)\int_{0}^{t}\frac{dy}{1+y^2}+\right.
\end{aligned}
$$

$$+\int_0^\infty \frac{\log((y+t)+2)}{\{1+y^2\}^{\frac{\nu+1}{2}}}dy\Bigg\}$$

$$\ll X^{-2}\left\{\int_0^\infty \frac{\log(x+t+2)}{1+(x+t)^2}dx+\log(t+2)\int_0^\infty \frac{dy}{1+y^2}+\right.$$

$$\left.+\int_0^\infty \frac{\log((y+t)+2)}{1+y^2}dy\right\}$$

$$\ll X^{-2}\left\{\int_t^\infty \frac{\log(y+2)}{1+y^2}dy+\log(t+2)\frac{\pi}{2}+\right.$$

$$\left.+\int_0^\infty \frac{\log((y+t)+2)}{1+y^2}dy\right\}$$

$$\ll X^{-2}\left\{\int_0^\infty \frac{\log(y+t+2)}{1+y^2}dy+\log(t+2)\right\}$$

$$\ll X^{-2}\left\{\int_0^{t+2} \frac{\log(y+t+2)}{1+y^2}dy+\int_{t+2}^\infty \frac{\log(y+t+2)}{1+y^2}dy+\right.$$

$$+\log(t+2)\}$$

$$\ll X^{-2}\left\{\int_0^{t+2} \frac{\log(y+t+2)}{1+y^2}dy+\int_{t+2}^\infty \frac{\log\{y(t+2)\}}{1+y^2}dy+\right.$$

$$+\log(t+2)\}$$

$$\ll X^{-2}\left\{\log\{2(t+2)\}\int_0^{t+2} \frac{dy}{1+y^2}+\int_{t+2}^\infty \frac{\log(t+2)}{1+y^2}dy+\right.$$

$$\left.+\int_{t+2}^\infty \frac{\log y}{1+y^2}dy+\log(t+2)\right\}$$

$$\ll X^{-2}\left\{\log\{2(t+2)\}\int_0^\infty \frac{dy}{1+y^2}+\log(t+2)\int_0^\infty \frac{dy}{1+y^2}+\right.$$

$$\left.+\int_{t+2}^\infty \frac{\log y}{1+y^2}dy+\log(t+2)\right\}$$

$$\ll X^{-2}\{\log\{2(t+2)\}+\log(t+2)+$$

$$\left.+\int_{t+2}^\infty \frac{\log y}{1+y^2}dy+\log(t+2)\right\}$$

$$\ll X^{-2}\left\{\log(t+2)+\int_{t+2}^\infty \frac{\log y}{1+y^2}dy\right\}$$

$$\ll X^{-2}\left\{\log(t+2)+\int_1^\infty \frac{\log y}{1+y^2}dy\right\}$$

$$= X^{-2}\left\{\log(t+2)+\mathbf{G}\right\}\ (\mathbf{G}=0.915964\cdots \text{は} \mathit{Catalan} \text{数}. [1])$$
$$\ll X^{-2}\log(t+2)\cdots(7)$$

$$\frac{1}{2\pi i}\int_{-\frac{\sigma-1}{2}-iT_m-it}^{-\frac{\sigma-1}{2}+iT_m-it} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw$$

については変数変換をしてから補題１を使う $\cdots(2)$：

$$\frac{1}{2\pi i}\int_{-\frac{\sigma-1}{2}-iT_m-it}^{-\frac{\sigma-1}{2}+iT_m-it} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw =$$
$$= \frac{(-1)^{\nu+1}}{2\pi i}\int_{\frac{\sigma-1}{2}-i(T_m-t)}^{\frac{\sigma-1}{2}+i(T_m+t)} -\frac{\zeta'}{\zeta}(s-z)\frac{X^{-z}}{z^{\nu+1}}dz$$
$$= (-1)^{\nu+1}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^s}\frac{1}{2\pi i}\int_{\frac{\sigma-1}{2}-i(T_m-t)}^{\frac{\sigma-1}{2}+i(T_m+t)} \frac{\left(\frac{n}{X}\right)^z}{z^{\nu+1}}dz$$
$$= \frac{(-1)^{\nu+1}}{\nu!}\sum_{n\ge X}\frac{\Lambda(n)}{n^s}\log^{\nu}\left(\frac{n}{X}\right)+$$
$$+(-1)^{\nu+1}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^s}\mathrm{O}\left(\frac{\left(\frac{n}{X}\right)^{\frac{\sigma-1}{2}}}{T_m^{\nu}(|\log\left(\frac{n}{X}\right)|+1)}\right)$$

($\sigma>1$ であった.)

$$= -\frac{1}{\nu!}\sum_{n\ge X}\frac{\Lambda(n)}{n^s}\log^{\nu}\left(\frac{X}{n}\right)+\mathrm{O}\left(\frac{1}{X^{\frac{\sigma-1}{2}}T_m^{\nu}}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^{\sigma-\frac{\sigma}{2}+\frac{1}{2}}}\right)$$
$$= -\frac{1}{\nu!}\sum_{n\ge X}\frac{\Lambda(n)}{n^s}\log^{\nu}\left(\frac{X}{n}\right)+\mathrm{O}\left(\frac{1}{X^{\frac{\sigma-1}{2}}T_m^{\nu}}\right)\cdots(8)$$

(3),(5),(6),(6’),(7),(8):

$$\frac{1}{2\pi i}\left\{\int_{-\frac{\sigma-1}{2}-iT_m-it}^{-\frac{\sigma-1}{2}+iT_m-it}+\int_{-\frac{\sigma-1}{2}+iT_m-it}^{(-1-\sigma)+iT_m-it}+\right.$$
$$\left.+\int_{(-1-\sigma)+iT_m-it}^{(-1-\sigma)-iT_m-it}+\int_{(-1-\sigma)-iT_m-it}^{-\frac{\sigma-1}{2}-iT_m-it}\right\}-\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw=$$
$$=\frac{X^{1-s}}{(1-s)^{\nu+1}}-\sum_{|\gamma|<T_m}\frac{X^{\rho-s}}{(\rho-s)^{\nu+1}}\cdots(3),(5),$$
$$\frac{1}{2\pi i}\int_{-\frac{\sigma-1}{2}+iT_m-it}^{(-1-\sigma)+iT_m-it}-\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw\ll$$
$$\ll X^{-\frac{\sigma-1}{2}}\frac{\log^2 T_m}{T_m^{\nu+1}}\ll\frac{\log^2 T_m}{T_m^{\nu+1}}\cdots(6),$$

$$\frac{1}{2\pi i}\int_{-1-\sigma+i(-T_m-t)}^{-\frac{\sigma-1}{2}+i(-T_m-t)} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw \ll \frac{\log^2 T_m}{T_m^{\nu+1}}\cdots(6'),$$
$$\frac{1}{2\pi i}\int_{(-1-\sigma)+iT_m-it}^{(-1-\sigma)-iT_m-it} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw \ll X^{-2}\log(t+2)\cdots(7),$$
$$\frac{1}{2\pi i}\int_{-\frac{\sigma-1}{2}-iT_m-it}^{-\frac{\sigma-1}{2}+iT_m-it} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw =$$
$$= -\frac{1}{\nu!}\sum_{n\geq X}\frac{\Lambda(n)}{n^s}\log^\nu\left(\frac{X}{n}\right) + \mathrm{O}\left(\frac{1}{X^{\frac{\sigma-1}{2}}T_m^\nu}\right)\cdots(8)$$

より

$$-\frac{1}{\nu!}\sum_{n\geq X}\frac{\Lambda(n)}{n^s}\log^\nu\left(\frac{X}{n}\right) + \mathrm{O}\left(\frac{1}{X^{\frac{\sigma-1}{2}}T_m^\nu}\right) +$$
$$+\mathrm{O}\left(X^{-\frac{\sigma-1}{2}}\frac{\log^2 T_m}{T_m^{\nu+1}}\right) + \mathrm{O}\left(X^{-2}\log(t+2)\right) + \mathrm{O}\left(X^{-\frac{\sigma-1}{2}}\frac{\log^2 T_m}{T_m^{\nu+1}}\right) =$$
$$= \frac{X^{1-s}}{(1-s)^{\nu+1}} - \sum_{|\gamma|<T_m}\frac{X^{\rho-s}}{(\rho-s)^{\nu+1}}\cdots(9)$$

(9) で $T_m\to\infty$ とすると

$$-\frac{1}{\nu!}\sum_{n\geq X}\frac{\Lambda(n)}{n^s}\log^\nu\left(\frac{X}{n}\right) + \mathrm{O}\left(X^{-2}\log(t+2)\right) =$$
$$= \frac{X^{1-s}}{(1-s)^{\nu+1}} - \sum_{\rho}\frac{X^{\rho-s}}{(\rho-s)^{\nu+1}}\cdots(10)(\Leftrightarrow(0))$$

これで証明は完了した。□.

[15] [16] [17] [18] [19] [20] [21] に証明なしで述べられている

Turán's explicit formula for primes in exact form

$s=\sigma+it\neq\rho,-2q,1\ (q=1,2,\cdots),\ \sigma>1,\ X>1,\ \nu\in\mathbf{N}$ に対して

$$\frac{1}{\nu!}\sum_{n\geq X}\frac{\Lambda(n)}{n^s}\log^\nu\left(\frac{X}{n}\right) =$$
$$= \sum_{\rho}\frac{X^{\rho-s}}{(\rho-s)^{\nu+1}} - \frac{X^{1-s}}{(1-s)^{\nu+1}} + \sum_{q=1}^{\infty}\frac{X^{(-2q)-s}}{((-2q)-s)^{\nu+1}}\cdots(11)$$

$\nu=\mathbf{0}$ の場合 **Landau** の明示公式 [5] がある。

この系 Corollary として

Turán's explicit formula for primes with error term

$s=\sigma+it,\ \sigma>1,\ X>1,\ \nu\in\mathbf{N}$ に対して,

$$\frac{1}{\nu!}\sum_{n\geq X}\frac{\Lambda(n)}{n^s}\log^{\nu}\left(\frac{X}{n}\right)=\sum_{\rho}\frac{X^{\rho-s}}{(\rho-s)^{\nu+1}}-\frac{X^{1-s}}{(1-s)^{\nu+1}}+\mathrm{O}\left(\frac{\log(|t|+2)}{X^2}\right)\cdots(0).$$

が直ちに導ける。

(11) からの (0) の証明

$$\sum_{q=1}^{\infty}\frac{X^{(-2q)-s}}{((-2q)-s)^{\nu+1}}\ll X^{-2}\ll\frac{\log(|t|+2)}{X^2}$$

を示せば良い。これは

$$\sum_{q=1}^{\infty}\frac{X^{(-2q)-s}}{((-2q)-s)^{\nu+1}}\ll X^{(-2\cdot 1)-\sigma}\ll X^{-3}$$

より明らか。これは (0) より良い評価である。$\square$.

(11) を証明するには次の補題が必要である。

補題 6 ([9]Lemma2 の一般化)
$s=\sigma+it\neq\rho,-2q,1$ (ρ は $\zeta(s)$ の複素零点, $q=1,2,\cdots$) に対して,

$$\begin{aligned}&\operatorname*{Res}_{w=0}\left\{-\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}\right\}\\ =\ &\frac{1}{\nu!}\frac{d^{\nu}}{dw^{\nu}}\left(-\frac{\zeta'}{\zeta}(s+w)X^w\right)\Bigg|_{w=0}\\ =\ &\sum_{n\leq X}\frac{\Lambda(n)}{n^s}\frac{1}{\nu!}\log^{\nu}\left(\frac{X}{n}\right)+\sum_{\rho}\frac{X^{\rho-s}}{(\rho-s)^{\nu+1}}+\sum_{q=1}^{\infty}\frac{X^{-2q-s}}{(-2q-s)^{\nu+1}}-\frac{X^{1-s}}{(1-s)^{\nu+1}}.\end{aligned}$$

補題 6 の証明
$\alpha:=\max\{2,1+\sigma\}$ として

$$\begin{aligned}&I_{\nu}:=\lim_{T_m\to\infty}I_{\nu}(m):=\\ &:=\lim_{T_m\to\infty}\frac{1}{2\pi i}\int_{\alpha-iT_m-it}^{\alpha+iT_m-it}-\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw\cdots(12)\end{aligned}$$

を考える。補題1で $T \to \infty$ の場合を適用して項別積分をすると

$$I_\nu = \frac{1}{\nu!}\sum_{n \le X} \frac{\Lambda(n)}{n^s} \log^\nu\left(\frac{X}{n}\right) \cdots (13)$$

$I_\nu(m)$ の積分路を次の鉤型の5つの積分路に移す Fig.2：

$$L_1 := (\alpha - i\infty, \alpha - iT_m - it],$$
$$L_2 := [\alpha - iT_m - it, -2m - 1 - \sigma - iT_m - it],$$
$$L_3 := [-2m - 1 - \sigma - iT_m - it, -2m - 1 - \sigma + iT_m - it],$$
$$L_4 := [-2m - 1 - \sigma + iT_m - it, \alpha + iT_m - it],$$
$$L_5 := [\alpha + iT_m - it, \alpha + i\infty).$$

即ち

$$\begin{aligned}
I_\nu(m) &= \frac{1}{2\pi i}\left\{\int_{L_1} + \int_{L_2} + \int_{L_3} + \int_{L_4} + \int_{L_5}\right\} - \frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw + \\
&+ \operatorname*{Res}_{w=0}\left\{-\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}\right\} + \sum_{-T_m - t < \gamma - t < T_m - t} \operatorname*{Res}_{w=\rho-s}\left\{-\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}\right\} + \\
&+ \sum_{2 \le 2q < 2m+1} \operatorname*{Res}_{w=-2q-s}\left\{-\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}\right\} + \operatorname*{Res}_{w=1-s}\left\{-\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}\right\} \\
&= \frac{1}{2\pi i}\left\{\int_{L_1} + \int_{L_2} + \int_{L_3} + \int_{L_4} + \int_{L_5}\right\} - \frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw + \\
&+ \frac{1}{\nu!}\frac{d^\nu}{dw^\nu}\left(-\frac{\zeta'}{\zeta}(s+w)X^w\right)\Bigg|_{w=0} - \sum_{|\gamma| < T_m} \frac{X^{\rho-s}}{(\rho-s)^{\nu+1}} - \\
&- \sum_{2 \le 2q < 2m+1} \frac{X^{-2q-s}}{(-2q-s)^{\nu+1}} + \frac{X^{1-s}}{(1-s)^{\nu+1}} \cdots (14)
\end{aligned}$$

(14) の中の5つの積分に補題5を適用すると ($\sigma > 0$ とする。)

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{L_1} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw = \\
&= \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha - iT_m - it} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw \ll \\
&\ll X^\alpha\left|-\frac{\zeta'}{\zeta}(\sigma+\alpha)\right|\int_{T_m}^{\infty}\frac{dv}{\alpha^2+v^2} \ll X^\alpha\left(\frac{\pi}{2} - \arctan T_m\right) \to 0 \\
&(as\ T_m \to \infty),
\end{aligned}$$

$$\frac{1}{2\pi i}\int_{L_2} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw =$$
$$= \frac{1}{2\pi i}\int_{\alpha-iT_m-it}^{-2m-1-\sigma-iT_m-it} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw \ll$$
$$\ll X^{\alpha}(2m+1+\sigma+\alpha)\log^2 m\frac{1}{T_m^{\nu+1}} \to 0 \ (as\ T_m \to \infty),$$
$$\frac{1}{2\pi i}\int_{L_3} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw =$$
$$= \frac{1}{2\pi i}\int_{-2m-1-\sigma-iT_m-it}^{-2m-1-\sigma+iT_m-it} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw \ll$$
$$\ll \int_{-T_m-t}^{T_m-t}\left|-\frac{\zeta'}{\zeta}(-2m-1+it+iv)\right|\frac{X^{-2m-1-\sigma}}{(2m+1+\sigma)^{\nu+1}}dv \ll$$
$$\ll 2T_m\cdot\log^2 m\frac{X^{-2m-1-\sigma}}{(2m+1+\sigma)^{\nu+1}} \to 0 \ (as\ m \to \infty),$$
$$\frac{1}{2\pi i}\int_{L_4} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw \ll$$
$$\ll X^{\alpha}(2m+1+\sigma+\alpha)\log^2 m\frac{1}{T_m^{\nu+1}} \to 0 \ (as\ T_m \to \infty),$$
$$\frac{1}{2\pi i}\int_{L_5} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw \ll X^{\alpha}\left(\frac{\pi}{2}-\arctan T_m\right) \to 0$$
$$(as\ T_m \to \infty).$$

これらの評価を (14) に適用すると (14) は

$$I_\nu(m) = \frac{1}{2\pi i}\left\{\int_{L_1}+\int_{L_2}+\int_{L_3}+\int_{L_4}+\int_{L_5}\right\} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw +$$
$$+\frac{1}{\nu!}\frac{d^\nu}{dw^\nu}\left(-\frac{\zeta'}{\zeta}(s+w)X^w\right)\bigg|_{w=0} - \sum_{|\gamma|<T_m}\frac{X^{\rho-s}}{(\rho-s)^{\nu+1}} -$$
$$-\sum_{2\le 2q<2m+1}\frac{X^{-2q-s}}{(-2q-s)^{\nu+1}} + \frac{X^{1-s}}{(1-s)^{\nu+1}} =$$
$$= \mathrm{O}\left(X^{\alpha}\left(\frac{\pi}{2}-\arctan T_m\right) + X^{\alpha}(2m+1+\sigma+\alpha)\log^2 m\frac{1}{T_m^{\nu+1}} +\right.$$
$$\left. +2T_m\cdot\log^2 m\frac{X^{-2m-1-\sigma}}{(2m+1+\sigma)^{\nu+1}}\right) +$$
$$+\frac{1}{\nu!}\frac{d^\nu}{dw^\nu}\left(-\frac{\zeta'}{\zeta}(s+w)X^w\right)\bigg|_{w=0} - \sum_{|\gamma|<T_m}\frac{X^{\rho-s}}{(\rho-s)^{\nu+1}} -$$

$$-\sum_{2\le 2q<2m+1}\frac{X^{-2q-s}}{(-2q-s)^{\nu+1}}+\frac{X^{1-s}}{(1-s)^{\nu+1}}\cdots(15)$$

となる。
ここで補題1を使うと

$$I_\nu(m)=\frac{1}{2\pi i}\int_{\alpha-iT_m-it}^{\alpha+iT_m-it}-\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w^{\nu+1}}dw=$$

$$=\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^s}\frac{1}{2\pi i}\int_{\alpha-iT_m-it}^{\alpha+iT_m-it}\frac{\left(\frac{X}{n}\right)^w}{w^{\nu+1}}dw=$$

$$=\frac{1}{\nu!}\sum_{n<X}\frac{\Lambda(n)}{n^s}\log^\nu\left(\frac{X}{n}\right)+\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^s}\mathrm{O}\left(\frac{\left(\frac{X}{n}\right)^\alpha}{T_m^\nu(|\log\left(\frac{X}{n}\right)|+1)}\right)=$$

$$=\frac{1}{\nu!}\sum_{n<X}\frac{\Lambda(n)}{n^s}\log^\nu\left(\frac{X}{n}\right)+\mathrm{O}\left(\frac{X^\alpha}{T_m^\nu}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^{\sigma+\alpha}}\right)=$$

$$=\frac{1}{\nu!}\sum_{n<X}\frac{\Lambda(n)}{n^s}\log^\nu\left(\frac{X}{n}\right)+\mathrm{O}\left(\frac{X^\alpha}{T_m^\nu}\right)$$

となるので (15) は

$$\frac{1}{\nu!}\sum_{n<X}\frac{\Lambda(n)}{n^s}\log^\nu\left(\frac{X}{n}\right)+\mathrm{O}\left(\frac{X^\alpha}{T_m^\nu}\right)=$$

$$=\mathrm{O}\left(X^\alpha\left(\frac{\pi}{2}-\arctan T_m\right)+X^\alpha(2m+1+\sigma+\alpha)\log^2 m\frac{1}{T_m^{\nu+1}}+\right.$$

$$\left.+2T_m\cdot\log^2 m\frac{X^{-2m-1-\sigma}}{(2m+1+\sigma)^{\nu+1}}\right)+$$

$$+\frac{1}{\nu!}\frac{d^\nu}{dw^\nu}\left(-\frac{\zeta'}{\zeta}(s+w)X^w\right)\Bigg|_{w=0}-\sum_{|\gamma|<T_m}\frac{X^{\rho-s}}{(\rho-s)^{\nu+1}}-$$

$$-\sum_{2\le 2q<2m+1}\frac{X^{-2q-s}}{(-2q-s)^{\nu+1}}+\frac{X^{1-s}}{(1-s)^{\nu+1}}\cdots(16)$$

となる。ここで $m\to\infty$ とすると (16) は

$$\frac{1}{\nu!}\sum_{n<X}\frac{\Lambda(n)}{n^s}\log^\nu\left(\frac{X}{n}\right)=$$

$$=\frac{1}{\nu!}\frac{d^\nu}{dw^\nu}\left(-\frac{\zeta'}{\zeta}(s+w)X^w\right)\Bigg|_{w=0}-\sum_{\rho}\frac{X^{\rho-s}}{(\rho-s)^{\nu+1}}-$$

$$-\sum_{q=1}^{\infty}\frac{X^{-2q-s}}{(-2q-s)^{\nu+1}}+\frac{X^{1-s}}{(1-s)^{\nu+1}}$$

となり，補題6が得られた。□.

(11) の別証明 another proof

$\Re s > 1$ とする。ここで $\frac{\zeta'}{\zeta}(s)$ は一様絶対収束する事に注意する。

$$
\begin{aligned}
&\frac{1}{\nu!}\frac{d^\nu}{dw^\nu}\left(-\frac{\zeta'}{\zeta}(s+w)X^w\right)\Bigg|_{w=0} = \\
&= \frac{1}{\nu!}\sum_{k=0}^{\nu}\binom{\nu}{k}\left\{-\frac{\zeta'}{\zeta}(s+w)^{(k)}\right\}(\log X)^{\nu-k}X^w\Bigg|_{w=0} \\
&= \frac{1}{\nu!}\sum_{k=0}^{\nu}\binom{\nu}{k}\left\{\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{s+w}}(-\log n)^k\right\}(\log X)^{\nu-k}X^w\Bigg|_{w=0} \\
&= \frac{1}{\nu!}\sum_{k=0}^{\nu}\binom{\nu}{k}\left\{\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{s}}(-\log n)^k\right\}(\log X)^{\nu-k} \\
&= \frac{1}{\nu!}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\sum_{k=0}^{\nu}\binom{\nu}{k}(-\log n)^k(\log X)^{\nu-k} \\
&= \frac{1}{\nu!}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}(\log X-\log n)^\nu \\
&= \frac{1}{\nu!}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\log^\nu\left(\frac{X}{n}\right)
\end{aligned}
$$

これを補題6の式に代入して (11) を得る。□.

附録 1. 補題 1 の証明

$\nu = 0$ の場合の証明を良く目にするが，$\nu \in \mathbf{N}$ の場合は，あまり目にする事はなく，[5] [6] に紹介されている場合より補題1の結果は良いので，ここに証明を掲げる。

$X > 1, \nu \in \{0\} \cup \mathbf{N}$ のとき

次の矩形の積分路を考える Fig.3：

$$
\begin{aligned}
&c > 0, b > 0, \\
&L_1 := [c-iT, c+iT], \\
&L_2 := [c+iT, -b+iT], \\
&L_3 := [-b+iT, -b-iT], \\
&L_4 := [-b-iT, c-iT].
\end{aligned}
$$

留数定理より

$$
\frac{1}{2\pi i}\left\{\int_{L_1}+\int_{L_2}+\int_{L_3}+\int_{L_4}\right\}\frac{X^w}{w^{\nu+1}}dw = \frac{(\log X)^\nu}{\nu!}\cdots(17)
$$

3つの積分

$$\frac{1}{2\pi i}\int_{L_2}\frac{X^w}{w^{\nu+1}}dw,\ \frac{1}{2\pi i}\int_{L_3}\frac{X^w}{w^{\nu+1}}dw,\ \frac{1}{2\pi i}\int_{L_4}\frac{X^w}{w^{\nu+1}}dw$$

を上から評価するが，これらが補題1の残余項となる：

$$\left|\frac{1}{2\pi i}\int_{L_2}\frac{X^w}{w^{\nu+1}}dw\right| \leq \frac{1}{2\pi}\int_{-b}^{c}\frac{X^u}{|u+iT|^{\nu+1}}du$$

$$\ll \frac{1}{T^{\nu+1}}\int_{-b}^{c}X^u du = \frac{1}{T^{\nu+1}}\left[\frac{X^u}{\log X}\right]\Big|_{u=-b}^{c} \leq \frac{X^c}{T^{\nu+1}\log X}\cdots(18),$$

L_4上の積分も同様に

$$\frac{1}{2\pi i}\int_{L_4}\frac{X^w}{w^{\nu+1}}dw \ll \frac{X^c}{T^{\nu+1}\log X}\cdots(19),$$

$$\left|\frac{1}{2\pi i}\int_{L_3}\frac{X^w}{w^{\nu+1}}dw\right| \leq \frac{1}{2\pi}\int_{-b-iT}^{-b+iT}\left|\frac{X^w}{w^{\nu+1}}\right||dw| \ll$$

$$\ll \frac{X^{-b}}{b^{\nu+1}}\int_{-T}^{T}dv = \frac{2TX^{-b}}{b^{\nu+1}} \to 0\ as\ b\to+\infty\cdots(20).$$

(17) で $b\to+\infty$ とすると (18),(19),(20) より

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{X^w}{w^{\nu+1}}dw + \mathrm{O}\left(\frac{X^c}{T^{\nu+1}\log X}\right) + 0 + \mathrm{O}\left(\frac{X^c}{T^{\nu+1}\log X}\right) =$$

$$= \frac{(\log X)^\nu}{\nu!},$$

従って

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{X^w}{w^{\nu+1}}dw = \frac{(\log X)^\nu}{\nu!} + \mathrm{O}\left(\frac{X^c}{T^{\nu+1}|\log X|}\right)\cdots(21)$$

$0<X<1, \nu\in\{0\}\cup\mathbf{N}$ のとき

次の矩形の積分路を考える Fig.4：

$$b>c>0,$$
$$L_1 := [c-iT, c+iT],$$
$$L_2 := [c+iT, b+iT],$$
$$L_3 := [b+iT, b-iT],$$
$$L_4 := [b-iT, c-iT].$$

再び留数定理より

$$\frac{1}{2\pi i}\left\{\int_{L_1}+\int_{L_2}+\int_{L_3}+\int_{L_4}\right\}\frac{X^w}{w^{\nu+1}}dw = 0\cdots(22)$$

やはり３つの積分

$$\frac{1}{2\pi i}\int_{L_2}\frac{X^w}{w^{\nu+1}}dw,\ \frac{1}{2\pi i}\int_{L_3}\frac{X^w}{w^{\nu+1}}dw,\ \frac{1}{2\pi i}\int_{L_4}\frac{X^w}{w^{\nu+1}}dw$$

を上から評価するが，これらが補題１の残余項となる:

$$\left|\frac{1}{2\pi i}\int_{L_2}\frac{X^w}{w^{\nu+1}}dw\right| \le \frac{1}{2\pi}\int_c^b \frac{X^u}{|u+iT|^{\nu+1}}du$$

$$\ll \frac{1}{T^{\nu+1}}\int_c^b X^u du = \frac{1}{T^{\nu+1}}\left[\frac{X^u}{\log X}\right]\Bigg|_{u=c}^{b} \le \frac{X^c}{T^{\nu+1}|\log X|}\cdots(23),$$

$(0<X<1$ を使った：$X^b<X^c.)$

L_4上の積分も同様に

$$\frac{1}{2\pi i}\int_{L_4}\frac{X^w}{w^{\nu+1}}dw \ll \frac{X^c}{T^{\nu+1}|\log X|}\cdots(24),$$

$$\left|\frac{1}{2\pi i}\int_{L_3}\frac{X^w}{w^{\nu+1}}dw\right| \le \frac{1}{2\pi}\int_{b-iT}^{b+iT}\left|\frac{X^w}{w^{\nu+1}}\right||dw| \ll$$

$$\ll \frac{X^b}{b^{\nu+1}}\int_{-T}^{T}dv = \frac{2TX^b}{b^{\nu+1}} \to 0\ as\ b\to+\infty\cdots(25).$$

(22) で $b\to+\infty$ とすると (23),(24),(25) より

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{X^w}{w^{\nu+1}}dw + \mathrm{O}\left(\frac{X^c}{T^{\nu+1}|\log X|}\right)+0+\mathrm{O}\left(\frac{X^c}{T^{\nu+1}|\log X|}\right)=$$

$$=0,$$

従って

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{X^w}{w^{\nu+1}}dw = 0+\mathrm{O}\left(\frac{X^c}{T^{\nu+1}|\log X|}\right)\cdots(26)$$

$X=1,\nu\in\mathbf{N}$ のとき

次の矩形の積分路を考える Fig.5：

$$c>0, R:=\sqrt{c^2+T^2}, \alpha:=\arctan\left(\frac{c}{T}\right)=\frac{c}{T}-\frac{1}{3}\left(\frac{c}{T}\right)^3+\cdots,$$

$$\frac{c}{T}<1,$$

$$L_1:=[c-iT,c+iT],$$

$$L_R:=\left\{Re^{i\theta}\ \middle|\ -\left(\frac{\pi}{2}-\alpha\right)\le\theta\le\frac{\pi}{2}-\alpha\right\}.$$

やはり留数定理より

$$\frac{1}{2\pi i}\left\{\int_{L_1}+\int_{L_R}\right\}\frac{1}{w^{\nu+1}}dw=0\cdots(27)$$

積分

$$\frac{1}{2\pi i}\int_{L_R}\frac{1}{w^{\nu+1}}dw$$

を評価する：

$$\left|\frac{1}{2\pi i}\int_{L_R}\frac{1}{w^{\nu+1}}dw\right|=\left|\frac{1}{2\pi i}\int_{-\left(\frac{\pi}{2}-\alpha\right)}^{\frac{\pi}{2}-\alpha}\frac{d(Re^{i\theta})}{(Re^{i\theta})^{\nu+1}}\right|=$$
$$=\frac{R}{2\pi R^{\nu+1}}\left|\int_{-\left(\frac{\pi}{2}-\alpha\right)}^{\frac{\pi}{2}-\alpha}\frac{ie^{i\theta}d\theta}{e^{i\theta(\nu+1)}}\right|=\frac{1}{2\pi R^{\nu}}\left|\int_{-\left(\frac{\pi}{2}-\alpha\right)}^{\frac{\pi}{2}-\alpha}e^{-i\nu\theta}d\theta\right|=$$
$$=\frac{1}{2\pi R^{\nu}}\left|\left[\frac{e^{-i\nu\theta}}{-i\nu}\right]\Bigg|_{\theta=-\left(\frac{\pi}{2}-\alpha\right)}^{\frac{\pi}{2}-\alpha}\right|=\frac{1}{2\pi\nu R^{\nu}}|e^{i\nu\left(\frac{\pi}{2}-\alpha\right)}-e^{-i\nu\left(\frac{\pi}{2}-\alpha\right)}|\ll$$
$$\ll\frac{1}{\nu R^{\nu}}\cdots(28)$$

(27),(28), $R=\sqrt{c^2+T^2}$ より

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{dw}{w^{\nu+1}}=0+\mathrm{O}\left(\frac{1}{\nu T^{\nu}}\right)\ for\ \nu\in\mathbf{N}\cdots(29)$$

$X=1,\nu=0$ のとき

留数定理を使わずに直接積分を評価する。$\frac{c}{T}<1$ を使って

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{dw}{w}=\frac{1}{2\pi i}\int_{-T}^{T}\frac{idv}{c+iv}=\frac{1}{2\pi i}\int_{-T}^{T}\frac{i(c-iv)dv}{c^2+v^2}=$$
$$=\frac{1}{2\pi i}\int_{-T}^{T}\frac{v}{c^2+v^2}dv+\frac{i}{2\pi i}\int_{-T}^{T}\frac{c}{c^2+v^2}dv=0+\frac{c}{2\pi}\int_{-T}^{T}\frac{dv}{c^2+v^2}=$$
$$=\frac{1}{2\pi}\int_{-\frac{T}{c}}^{\frac{T}{c}}\frac{dy}{1+y^2}=\frac{1}{\pi}\int_{0}^{\frac{T}{c}}\frac{dy}{1+y^2}=\frac{1}{\pi}\left\{\int_0^{\infty}\frac{dy}{1+y^2}-\int_{\frac{T}{c}}^{\infty}\frac{dy}{1+y^2}\right\}=$$
$$=\frac{1}{\pi}\left\{\frac{\pi}{2}-\int_{\frac{T}{c}}^{\infty}\frac{dy}{1+y^2}\right\}=\frac{1}{2}-\frac{1}{\pi}\int_{\frac{T}{c}}^{\infty}\frac{dy}{1+y^2}=$$
$$=\frac{1}{2}-\frac{1}{\pi}\left\{\int_0^{\infty}\frac{dy}{1+y^2}-\int_0^{\frac{T}{c}}\frac{dy}{1+y^2}\right\}=\frac{1}{2}-\frac{1}{\pi}\left\{\frac{\pi}{2}-\arctan\left(\frac{T}{c}\right)\right\}=$$
$$=\frac{1}{2}-\frac{1}{\pi}\left\{\frac{\pi}{2}-\left(\frac{\pi}{2}-\arctan\left(\frac{c}{T}\right)\right)\right\}=\frac{1}{2}-\frac{1}{\pi}\arctan\left(\frac{c}{T}\right)=$$

$$= \frac{1}{2} - \frac{1}{\pi}\left\{\left(\frac{c}{T}\right) - \frac{1}{3}\left(\frac{c}{T}\right)^3 + \cdots\right\} =$$
$$= \frac{1}{2} - \frac{1}{\pi}\left(\frac{c}{T}\right) + \mathrm{O}\left(\left(\frac{c}{T}\right)^3\right) = \frac{1}{2} + \mathrm{O}\left(\frac{c}{T}\right) \cdots (30)$$

(21),(26),(29),(30) で補題1の証明の大半は完了した。

しかし，補題1の O-terms の中の部分を証明せねばならない。これらの項が重要なのは $|\log X|$ が0に近い場合，即ち $1 \neq X \sim 1$ の場合である。従って $\frac{c}{T} < 1, T \gg 1$ であるから，$X = 1$ の場合は考える必要はない。

$X > 1, \nu \in \{0\} \cup \mathbf{N}$ のとき
積分路を次で定める Fig.6:

$$c > 0, R := \sqrt{c^2 + T^2}, \alpha := \arctan\left(\frac{c}{T}\right) = \frac{c}{T} - \frac{1}{3}\left(\frac{c}{T}\right)^3 + \cdots,$$
$$\frac{c}{T} < 1,$$
$$L_1 := [c - iT, c + iT],$$
$$L_R := \left\{Re^{i\theta} \mid \frac{\pi}{2} - \alpha \leq \theta \leq \frac{3\pi}{2} + \alpha\right\}.$$

留数定理より

$$\frac{1}{2\pi i}\left\{\int_{L_1} + \int_{L_R}\right\}\frac{X^w}{w^{\nu+1}}dw = \frac{(\log X)^\nu}{\nu!} \cdots (31)$$

積分

$$\frac{1}{2\pi i}\int_{L_R}\frac{X^w}{w^{\nu+1}}dw$$

を評価する：

$$\left|\frac{1}{2\pi i}\int_{L_R}\frac{X^w}{w^{\nu+1}}dw\right| = \left|\frac{1}{2\pi i}\int_{\frac{\pi}{2}-\alpha}^{\frac{3\pi}{2}+\alpha}\frac{X^{Re^{i\theta}}d(Re^{i\theta})}{(Re^{i\theta})^{\nu+1}}\right| =$$
$$= \frac{R}{2\pi R^{\nu+1}}\left|\int_{\frac{\pi}{2}-\alpha}^{\frac{3\pi}{2}+\alpha}\frac{X^{Re^{i\theta}}ie^{i\theta}d\theta}{e^{i\theta(\nu+1)}}\right| \leq \frac{1}{2\pi R^\nu}\int_{\frac{\pi}{2}-\alpha}^{\frac{3\pi}{2}+\alpha}X^{R\cos\theta}d\theta \leq$$
$$\leq \frac{X^c}{2\pi R^\nu}\int_{\frac{\pi}{2}-\alpha}^{\frac{3\pi}{2}+\alpha}d\theta \ll \frac{X^c}{R^\nu} \cdots (32)$$

(31),(32), $R = \sqrt{c^2 + T^2}$ より

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{X^w dw}{w^{\nu+1}} = \frac{(\log X)^\nu}{\nu!} + \mathrm{O}\left(\frac{X^c}{T^\nu}\right) \;\; for\ \nu \in \{0\} \cup \mathbf{N} \cdots (33)$$

(21),(33) を合せ考え
$X > 1, \nu \in \{0\} \cup \mathbf{N}$ のとき

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{X^w}{w^{\nu+1}}dw = $$
$$= \frac{(\log X)^\nu}{\nu!} + \mathrm{O}\left(\frac{X^c}{T^\nu}\min\left\{1, \frac{1}{T|\log X|}\right\}\right)\cdots(34)$$

$0 < X < 1, \nu \in \{0\} \cup \mathbf{N}$ のとき
積分路を次で定める Fig.7:

$$c > 0, R := \sqrt{c^2 + T^2}, \alpha := \arctan\left(\frac{c}{T}\right) = \frac{c}{T} - \frac{1}{3}\left(\frac{c}{T}\right)^3 + \cdots,$$
$$\frac{c}{T} < 1,$$
$$L_1 := [c - iT, c + iT],$$
$$L_R := \left\{Re^{i\theta} \mid -\left(\frac{\pi}{2} - \alpha\right) \le \theta \le \frac{\pi}{2} - \alpha\right\}.$$

留数定理より

$$\frac{1}{2\pi i}\left\{\int_{L_1} + \int_{L_R}\right\}\frac{X^w}{w^{\nu+1}}dw = 0\cdots(35)$$

積分

$$\frac{1}{2\pi i}\int_{L_R}\frac{X^w}{w^{\nu+1}}dw$$

を評価する：

$$\left|\frac{1}{2\pi i}\int_{L_R}\frac{X^w}{w^{\nu+1}}dw\right| = \left|\frac{1}{2\pi i}\int_{-(\frac{\pi}{2}-\alpha)}^{\frac{\pi}{2}-\alpha}\frac{X^{Re^{i\theta}}d(Re^{i\theta})}{(Re^{i\theta})^{\nu+1}}\right| =$$
$$= \frac{R}{2\pi R^{\nu+1}}\left|\int_{-(\frac{\pi}{2}-\alpha)}^{\frac{\pi}{2}-\alpha}\frac{X^{Re^{i\theta}}ie^{i\theta}d\theta}{e^{i\theta(\nu+1)}}\right| \le \frac{1}{2\pi R^\nu}\int_{-(\frac{\pi}{2}-\alpha)}^{\frac{\pi}{2}-\alpha}X^{R\cos\theta}d\theta \le$$
$$\le \frac{X^c}{2\pi R^\nu}\int_{-(\frac{\pi}{2}-\alpha)}^{\frac{\pi}{2}-\alpha}d\theta \ll \frac{X^c}{R^\nu}\cdots(36)$$

(35),(36), $R = \sqrt{c^2 + T^2}$ より

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{X^w dw}{w^{\nu+1}} = 0 + \mathrm{O}\left(\frac{X^c}{T^\nu}\right) \ for\ \nu \in \{0\} \cup \mathbf{N}\cdots(37)$$

(26),(37) を合せ考え
$0 < X < 1, \nu \in \{0\} \cup \mathbf{N}$ のとき

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{X^w}{w^{\nu+1}}dw =$$
$$= 0 + \mathrm{O}\left(\frac{X^c}{T^\nu}\min\left\{1, \frac{1}{T|\log X|}\right\}\right) \cdots (38)$$

(29),(30),(34),(38) より補題 1 は完全に証明された。□

附録 2.Landau の明示公式 explicit formula:$\nu = 0$ の場合の証明

次の積分路を考える：Fig.8

$\sigma \geq 0,\ \alpha + \sigma > 1,\ T \gg 1,\ T \gg t,\ \mathbf{N} \ni l \gg 1,\ t \gg 1,\ X > 1,$
補題 2 の T_m を T とする。

$$L_1 := [\alpha - iT - it, \alpha + iT - it],$$
$$L_2 := [\alpha + iT - it, -2l - 1 - \sigma + iT - it],$$
$$L_3 := [-2l - 1 - \sigma + iT - it, -2l - 1 - \sigma - iT - it],$$
$$L_4 := [-2l - 1 - \sigma - iT - it, \alpha - iT - it].$$

留数定理より

$$\frac{1}{2\pi i}\int_{L} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w}dw :=$$
$$:= \frac{1}{2\pi i}\left\{\int_{L_1} + \int_{L_2} + \int_{L_3} + \int_{L_4}\right\} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w}dw :=$$
$$:= \frac{1}{2\pi i}\left\{\int_{\alpha-iT-it}^{\alpha+iT-it} + \int_{\alpha+iT-it}^{-2l-1-\sigma+iT-it} + \right.$$
$$\left. + \int_{-2l-1-\sigma+iT-it}^{-2l-1-\sigma-iT-it} + \int_{-2l-1-\sigma-iT-it}^{\alpha-iT-it}\right\} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w}dw =$$
$$= -\frac{\zeta'}{\zeta}(s) + \frac{X^{1-s}}{1-s} - \sum_{|\Im\rho|<T}\frac{X^{\rho-s}}{\rho - s} - \sum_{q=1}^{l}\frac{X^{-2q-s}}{-2q-s} \cdots (39)$$

上記 3 つの積分

$$\frac{1}{2\pi i}\int_{L_2} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w}dw,$$
$$\frac{1}{2\pi i}\int_{L_3} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w}dw,$$
$$\frac{1}{2\pi i}\int_{L_4} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w}dw$$

を上から評価する：

$$\frac{1}{2\pi i}\int_{L_2} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w}dw =$$
$$= \frac{1}{2\pi i}\int_{\alpha+iT-it}^{-2l-1-\sigma+iT-it} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w}dw \ll$$
$$\ll \int_{-2l-1-\sigma}^{\alpha}\left|-\frac{\zeta'}{\zeta}(\sigma+it+u+iT-it)\right|\frac{X^u}{T}du \ll$$
$$\ll \int_{-2l-1-\sigma}^{\alpha}\frac{(\log T)^2 X^u}{T}du \ll \frac{(\log T)^2}{T}\int_{-2l-1-\sigma}^{\alpha} X^u du =$$
$$= \frac{(\log T)^2}{T}\left[\frac{X^u}{\log X}\right]\Bigg|_{u=\alpha}^{-2l-1-\sigma} \ll \frac{(\log T)^2}{T}\left\{\frac{X^\alpha}{\log X}\right\}\cdots(40)$$

同様にして

$$\frac{1}{2\pi i}\int_{L_4} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w}dw =$$
$$= \frac{1}{2\pi i}\int_{-2l-1-\sigma-iT-it}^{\alpha-iT-it} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w}dw \ll$$
$$\ll \frac{(\log T)^2}{T}\left\{\frac{X^\alpha}{\log X}\right\}\cdots(41)$$

補題3を使って

$$\frac{1}{2\pi i}\int_{L_3} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w}dw =$$
$$= \frac{1}{2\pi i}\int_{-2l-1-\sigma-iT-it}^{-2l-1-\sigma+iT-it} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w}dw \ll$$
$$\ll \int_{-T-t}^{T-t}\left|-\frac{\zeta'}{\zeta}(\sigma+it+(-2l-1-\sigma)+iv)\right|\frac{X^{-2l-1-\sigma}}{2l+1+\sigma}dv \ll$$
$$\ll \frac{X^{-2l-1-\sigma}}{2l+1+\sigma}\int_{-T-t}^{T-t}\log(|-2l-1+i(t+v)|+1)dv \ll$$
$$\ll 2T\log(|-2l-1+iT|+1)\frac{X^{-2l-1-\sigma}}{2l+1+\sigma} \to 0 \ as\ l \to +\infty.$$
$$\cdots(42)$$

(39) で $l \to +\infty$ として，(40),(41),(42) を使うと (39) は

$$\frac{1}{2\pi i}\int_{\alpha-iT-it}^{\alpha+iT-it} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w}dw +$$
$$+\mathrm{O}\left(\frac{(\log T)^2}{T}\left\{\frac{X^\alpha}{\log X}\right\}\right) + 0 + \mathrm{O}\left(\frac{(\log T)^2}{T}\left\{\frac{X^\alpha}{\log X}\right\}\right) =$$

$$= -\frac{\zeta'}{\zeta}(s) + \frac{X^{1-s}}{1-s} - \sum_{|\Im\rho|<T} \frac{X^{\rho-s}}{\rho-s} - \sum_{q=1}^{\infty} \frac{X^{-2q-s}}{-2q-s} \cdots (43)$$

となる。(43) に於いて $T \to \infty$ とすると

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w}dw =$$

$$= -\frac{\zeta'}{\zeta}(s) + \frac{X^{1-s}}{1-s} - \lim_{T\to\infty}\sum_{|\Im\rho|<T} \frac{X^{\rho-s}}{\rho-s} - \sum_{q=1}^{\infty} \frac{X^{-2q-s}}{-2q-s} \cdots (44)$$

(44) の左辺の積分で $\Re(\alpha+s) = \alpha+\sigma > 1$ であるので被積分関数の中の $-\frac{\zeta'}{\zeta}$ を Dirichlet 級数に展開して，これは絶対一様収束するので項別積分が出来，補題 1 を適用出来る：

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} -\frac{\zeta'}{\zeta}(s+w)\frac{X^w}{w}dw = \sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^s}\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{\left(\frac{X}{n}\right)^w}{w}dw =$$

$$= {\sum_{n\le X}}' \frac{\Lambda(n)}{n^s}$$

${\sum_{n\le X}}'$ は count の際 (n を加える際)，$n = X \in \mathbf{N}$ の項に対しては $\frac{1}{2}$ を乗ずる.

従って

Landau の明示公式 explicit formula:

$${\sum_{n\le X}}' \frac{\Lambda(n)}{n^s} = -\frac{\zeta'}{\zeta}(s) + \frac{X^{1-s}}{1-s} - \lim_{T\to\infty}\sum_{|\Im\rho|<T} \frac{X^{\rho-s}}{\rho-s} - \sum_{q=1}^{\infty} \frac{X^{-2q-s}}{-2q-s}. \cdots (45)$$

を得る。これから

Turán 型の明示公式 explicit formula：

$${\sum_{n\ge X}}' \frac{\Lambda(n)}{n^s} = -\frac{X^{1-s}}{1-s} + \lim_{T\to\infty}\sum_{|\Im\rho|<T} \frac{X^{\rho-s}}{\rho-s} + \sum_{q=1}^{\infty} \frac{X^{-2q-s}}{-2q-s} \ for\ \sigma > 1. \cdots (46)$$

を得る為には，(45) で $\Re s > 1$ として $-\frac{\zeta'}{\zeta}(s)$ を Dirichlet 級数に展開する：

$${\sum_{n\le X}}' \frac{\Lambda(n)}{n^s} =$$

$$= \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s} + \frac{X^{1-s}}{1-s} - \lim_{T\to\infty} \sum_{|\Im\rho|<T} \frac{X^{\rho-s}}{\rho-s} - \sum_{q=1}^{\infty} \frac{X^{-2q-s}}{-2q-s}$$

$$\Longleftrightarrow$$

$$-{\sum_{n\geq X}}' \frac{\Lambda(n)}{n^s} = \frac{X^{1-s}}{1-s} - \lim_{T\to\infty} \sum_{|\Im\rho|<T} \frac{X^{\rho-s}}{\rho-s} - \sum_{q=1}^{\infty} \frac{X^{-2q-s}}{-2q-s}$$

(43) の積分に直接，補題 1 の $\nu = 0$ の場合を適用すると

$$\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s} \frac{1}{2\pi i} \left\{ \int_{\alpha-i(T+t)}^{\alpha+i(T+t)} - \int_{\alpha+i(T-t)}^{\alpha+i(T+t)} \right\} \frac{\left(\frac{X}{n}\right)^w}{w} dw +$$
$$+\mathrm{O}\left(\frac{(\log T)^2}{T} \left\{ \frac{X^\alpha}{\log X} \right\} \right) =$$
$$= -\frac{\zeta'}{\zeta}(s) + \frac{X^{1-s}}{1-s} - \sum_{|\Im\rho|<T} \frac{X^{\rho-s}}{\rho-s} - \sum_{q=1}^{\infty} \frac{X^{-2q-s}}{-2q-s} \cdots (43')$$

$$\frac{1}{2\pi i} \int_{\alpha-iT-it}^{\alpha+iT-it} -\frac{\zeta'}{\zeta}(s+w) \frac{X^w}{w} dw =$$
$$= \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s} \frac{1}{2\pi i} \left\{ \int_{\alpha-i(T+t)}^{\alpha+i(T+t)} - \int_{\alpha+i(T-t)}^{\alpha+i(T+t)} \right\} \frac{\left(\frac{X}{n}\right)^w}{w} dw =$$
$$= \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s} \frac{1}{2\pi i} \int_{\alpha-i(T+t)}^{\alpha+i(T+t)} \frac{\left(\frac{X}{n}\right)^w}{w} dw -$$
$$- \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s} \frac{1}{2\pi i} \int_{\alpha+i(T-t)}^{\alpha+i(T+t)} \frac{\left(\frac{X}{n}\right)^w}{w} dw =$$
$$= \sum_{n<X} \frac{\Lambda(n)}{n^s} \left\{ 1 + \mathrm{O}\left(\frac{\left(\frac{X}{n}\right)^\alpha}{T|\log\left(\frac{X}{n}\right)|} \right) \right\} + \delta_X \frac{\Lambda(X)}{X^s} \left\{ \frac{1}{2} + \mathrm{O}\left(\frac{\alpha}{T}\right) \right\} +$$
$$+ \sum_{n>X} \frac{\Lambda(n)}{n^s} \left\{ 0 + \mathrm{O}\left(\frac{\left(\frac{X}{n}\right)^\alpha}{T|\log\left(\frac{X}{n}\right)|} \right) \right\} +$$
$$+\mathrm{O}\left(\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^\sigma} \int_{\alpha+i(T-t)}^{\alpha+i(T+t)} \frac{\left(\frac{X}{n}\right)^\alpha}{T-t} dw \right) =$$

$$\textit{where}\ \ \delta_X := \begin{cases} 1\,, & (X \in \mathbf{N}) \\ 0\,, & (X \notin \mathbf{N}), \end{cases}$$

$$= {\sum_{n\leq X}}' \frac{\Lambda(n)}{n^s} + \mathrm{O}\left(\frac{X^\alpha}{T} \sum_{n<X} \frac{\Lambda(n)}{n^{\sigma+\alpha}|\log\left(\frac{X}{n}\right)|} \right) + \mathrm{O}\left(\delta_X \frac{\alpha\Lambda(X)}{TX^\sigma} \right) +$$

$$+\mathrm{O}\left(\frac{X^\alpha}{T}\sum_{n>X}\frac{\Lambda(n)}{n^{\sigma+\alpha}|\log\left(\frac{X}{n}\right)|}\right)+\mathrm{O}\left(\frac{X^\alpha}{T-t}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^{\sigma+\alpha}}\cdot 2t\right)=$$

$$=\sum_{n\le X}{}'\frac{\Lambda(n)}{n^s}+\mathrm{O}\left(\frac{X^\alpha}{T}\cdot 1\right)+\mathrm{O}\left(\delta_X\frac{\alpha\Lambda(X)}{TX^\sigma}\right)+$$

$$+\mathrm{O}\left(\frac{tX^\alpha}{T}\cdot 1\right)+\mathrm{O}\left(\frac{X^\alpha}{T-t}\cdot 1\right)=$$

$$=\sum_{n\le X}{}'\frac{\Lambda(n)}{n^s}+\mathrm{O}_{s,\alpha,X}\left(\frac{1}{T}\right)\cdots\ (47)$$

(43’) 又は (43) に (47) を代入して

主定理 1 *Main Theorem*

$$\sum_{n\le X}{}'\frac{\Lambda(n)}{n^s}+\mathrm{O}_{s,\alpha,X}\left(\frac{1}{T}\right)+\mathrm{O}\left(\frac{(\log T)^2}{T}\left\{\frac{X^\alpha}{\log X}\right\}\right)=$$

$$=-\frac{\zeta'}{\zeta}(s)+\frac{X^{1-s}}{1-s}-\sum_{|\Im\rho|<T}\frac{X^{\rho-s}}{\rho-s}-\sum_{q=1}^{\infty}\frac{X^{-2q-s}}{-2q-s}\cdots(48)$$

を得る。

参考文献

[1] Gradstein,I.G., Rijik,I.M.: *Tables of Integrals, Sums, Series and Products* (in Russian), Nauka, Moscow, 1971.
(邦訳：大槻義彦 訳『数学大公式集』，丸善，1983, (xviii+1085)pp., p.532, 4.231.)

[2] Ingham,A.E.: *The Distribution of Prime Numbers*, Cambridge Univ. Press, 1932.

[3] Karatsuba,A.A. (translated by Nathanson,M.B.): *Basic Analytic Number Theory*, Springer, Berlin, Heidelberg, 1993, traslation from the original Russian ed.: Osnovy analiticheskoj teorii chisel, 2nd ed., Nauka, Moscow 1983.

[4] Kowalski,E.: *Un cours de théorie analytique des nombres*, Société Mathématique de France, 2004.

[5] Landau,E.: *Handbuch der Leher von der Verteilung der Primzahlen*, Teubner, Leipzig, 1909, § 66,87,88. [Reprint: Chelsea 1953, Chelsea 1974, two volumes in one.]

[6] 三井孝美 Mitsui,T.:『整数論；解析的整数論入門』至文堂，1970. *Number Theory; Introduction to Analytic Number Theory* (in Japanese), Shibunndo, Tokyo, 1970.

[7] Murty,M.Ram : *Problems in Analytic Number Theory*, Springer, New York, 2001, (xvi+452)pp.

[8] Narkiewicz,W.: *The Developmennt of Prime Number Theory*, Springer, Berlin, Heidelberg, 2000.
(邦訳：中嶋眞澄 訳『素数定理の進展・上』，丸善出版，訂正版 2014, 2012,『素数定理の進展・下』，丸善出版，2013)

[9] Selberg,A.: On the normal density of primes in small intervals, and the difference between consecutive primes, *Arch. f. Math. og Naturv.*, **B. 47**(1943), No.6, 87-105.

[10] 龍沢周雄 Tatuzawa,T.:『関数論』共立出版，共立全書 233，1980. *Complex Function Theory* (in Japanese), Kyouritsu-Shuppan, Tokyo, 1980.

[11] Titchmarsh,E.C.: *The Theory of the Riemann Zeta-function*, Clarendon Press, Oxford, 1951. [2nd ed. revised by Heath-Brown,D.R., 1986.]

[12] Turán,P. (ハンガリー語では姓・名の順となる): *Eine Neue Methode in der Analysis und deren Anwendungenn*, Budapest Akad. Kiadó, 1953.

[13] Turán,P. 保羅杜瀾:『数学分析中的一個新方法及其應用』数学進展 (3 期)，第 2 巻 (1956)，309-565，(vii+251)pp., p.487, 輔助定理 21.1 (Chinese ed. translated and revised from [12].)

[14] Turán,P.: *On A New Method of Analysis and its Applications*, Wiley-Interscience, John Wiley & Sons, 1984, (xvi+584)pp., p.567,568, Appendix F, Lemma F.1. (translated and revised from [12] [13].)

[15] * Turán,P.: On Riemann's Hypothesis, Izv. Akad. Nauk SSSR **11**(1947), 179-262.
(306-368, p.314 (1.11.1), p.318 Lemma IV, p.332 footnote.)

[16] ** Turán,P.: On the remainder term of the prime-number formula II, AMASH**1**(1950), 48-63.
(541-551, p.547,548.)

[17] ** Turán,P.: On Carlson's theorem in the theory of zeta-function of Riemann, AMASH**2**(1951), 39-73. (584-617, p.589)

[18] ** Turán,P.: On the zeros of Riemann's function, MTA III. Oszt. Közl. **4**(1954), 357-368, (in Hugarian). (748-759, p.754(34).)

[19] ** Turán,P.: On Lindelöf's conjecture, AMASH**5**(1954), 145-163. (765-783, p.776(10.3).)

[20] *** Turán,P.: On the zeros of the zeta-function of Riemann, J. Indian Math. Soc. **20**(1956), 17-36. (890-909, p.897(4.3))

[21] ** Turán,P.: Über Eine Anwendung Einer Neuen Methode auf die Theorie der Riemannschen Zetefunktion, Wiss. Z. Humboldt-Univ. Berlin Math.-Natur Reihe 1955/1956, 275-279. (923-930, p.929(18).)
*···exact formula with proof in case of $\nu = 1$,
**···exact formula without proof,
***···formula with error term & without proof.

(received 4 July 2022.)

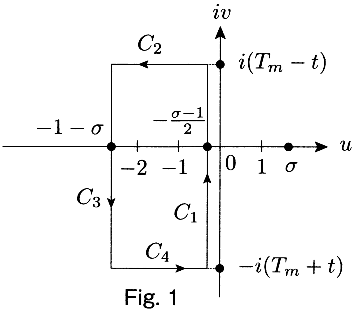

Fig. 1

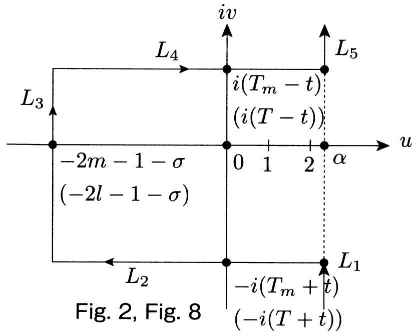

Fig. 2, Fig. 8

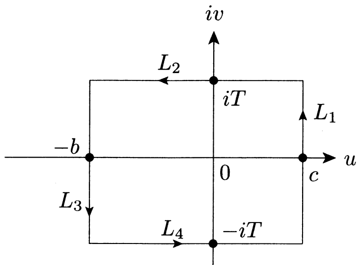

Fig. 3

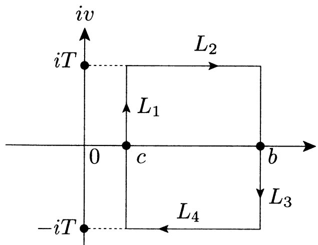

Fig. 4

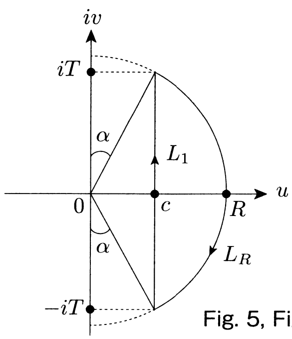

Fig. 5, Fig. 7

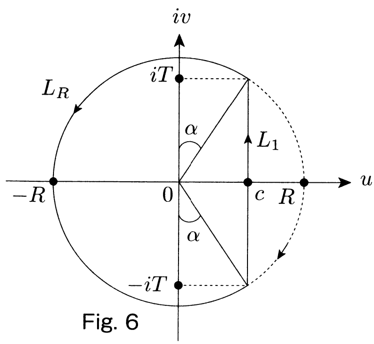

Fig. 6